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THE STUDY OF WHISPERING MODES IN ANISOTROPIC AND ISOTROPIC DIELECTRIC SPHERICAL RESONATORS

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Abstract - **The Frequency Standards and Metrology Research Group at the University of Western Australia was recently funded to perform a new Michelson-Morley (MM) experiment using a low noise Dual-Mode oscillator, based on a spherical microwave resonator [1]. One option is to use a high-Q dielectrically loaded cavity. The mode structure, frequency and Q-factor of Whispering modes within spherical dielectric resonators have been investigated. In particular we examined the frequency degenerate Whispering Spherical mode families and discuss their potential to form the basis of a sensitive MM test in the microwave regime. It is possible to solve a spherically-symmetrical system analytically [2,3] but, in practice, departures from perfect sphericity occur due to the necessity of a support structure for the dielectric and hence it is necessary to implement a numerical solution. Finite element modeling [4] is used to predict the mode structure of two spherical copper resonators, the first loaded with an isotropic fused-silica sphere loaded with cylindrical Teflon supports, and the other with a HEMEX sapphire "sphere on a stick". Predicted frequency splitting of the degenerate modes is confirmed by comparing experiment with finite element calculations, and is shown to be within experimental error set by the dimensions of the resonators.**

***Keywords* - microwave , resonator, spherical**

I. INTRODUCTION

Whispering Gallery (WG) modes in cylindrical systems can be visualized as a superposition of two rays, one moving clockwise the other anticlockwise, propagating around the cylinder in an integer number of reflections. These modes have been studied for some time, and have found applications as high Q resonators and filters in the microwave regime. Spherical systems possess similar modes that "whisper" around the surface of the sphere. We call these modes Whispering Spherical (WS) modes. It has been recently suggested that these modes could be used to perform a Michelson-Morley experiment [1], or be used in the development of high Q multipole filters in a spherical cavity. [5]

WS modes arise as frequency degenerate sets. Each member of the set has a distinct field structure and propagation pattern however, in order to examine and utilize individual modes some method of breaking the degeneracy must be found. Furthermore, it is advantageous if it is done in a way that allows for the easy identification of each mode. One method to achieve this is via the introduction of a perturbation to the system that is cylindrically symmetric with respect to the polar axis. The perturbation breaks the degeneracy of the WS mode families, introducing frequency shifts that can be reliably predicted using finite element modeling techniques. In this work two near-spherical dielectrically loaded copper cavity resonators are examined. One cavity was loaded with an isotropic fused-silica sphere, the other with a HEMEX sapphire "sphere on a stick."

II. SOLVING ISOTROPIC SYSTEMS

Applying a separation of variables technique to Maxwell's equations in spherical polar coordinates (r, θ, ϕ) leads to the general solution of the radial electric and magnetic field components, E_r and H_r, which satisfy the resonance condition [2,3]. The mode solutions are classed as Transverse Magnetic (TM) or Transverse Electric (TE) based on whether the magnetic or electric field is aligned transverse to the radial direction. General solutions for E_r and H_r are given by (1) and (2) for TM and TE modes respectively:

$$E_r = n(n+1)\frac{\sqrt{k_p r}}{r^2} J_{n+\frac{1}{2}}\left(k_p r\right) P_n^m\left(Cos\theta\right) \begin{matrix} Cos(m\phi) \\ Sin(m\phi) \end{matrix} \quad H_r = 0 \tag{1}$$

$$H_r = n(n+1)\frac{\sqrt{k_p r}}{r^2} J_{n+\frac{1}{2}}\left(k_p r\right) P_n^m\left(Cos\theta\right) \begin{matrix} Cos(m\phi) \\ Sin(m\phi) \end{matrix} \quad E_r = 0 \tag{2}$$

Where n is the mode number, k_p is the wave number and m is the azimuthal mode number, the number of 2π variations in the ϕ direction, which varies from 0 to n. The wave number takes discrete values corresponding to an integer number of variations in the radial direction, denoted by p. Modes that correspond to p = 1 are the WS modes. In this work TM and TE modes are denoted TM_{npm} and TE_{npm} respectively.

The other field components (E_θ, H_θ, E_ϕ, H_ϕ) can be determined from E_r and H_r by applying Maxwell's equations. As one would expect the θ and ϕ terms are cyclic in 2π and are hence common to all spherical systems.

In order to find a specific solution the composition of the system must be taken into account by solving the radial boundary value problem. In the case of a spherically-symmetric dielectrically-loaded cavity, the dimensions of the system, the permittivity of the dielectric and the shielding effect of the cavity walls set limits on the radial component which leads to the complete solution.

III. Degeneracy

The mode frequency in a spherical system is dependent on n and p, but independent of m. Hence a 2n + 1 frequency degeneracy arises from the θ and φ terms. The Legendre polynomial term contributes n + 1 distinct mode patterns plus a further two-fold degeneracy for modes with m ≠ 0 from the sin(mφ) or cos(mφ) term. Each degenerate set of 2n + 1 modes is known as a mode family. The sin/cos splitting represents a 90º phase shift of the entire field pattern in the φ direction. Apart from that, the modes are identical. This means that the two modes are interchangeable for any practical application. Therefore this work concerns itself only with the Legendre degeneracy. Field density plots for the TM_{41m} mode family in an empty cavity are given in Figure 1 as an aid to visualizing the Legendre mode patterns.

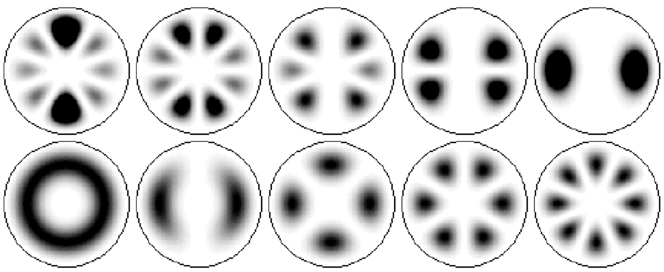

Figure 1: Typical radial magnetic field density plots for the TE_{41m} mode family in a spherical system. The top row shows a cross section through the y-z (meridian) plane and the bottom row through the x-y (equatorial) plane. The m = n = 4 WG mode (far right) propagates almost entirely in the azimuthal direction with no θ component. The m = 0 mode (far left) propagates as a circular wave front in the θ direction with no φ component. The other modes are essentially linear combinations of these two extremes.

The WS mode with m = n is a WG mode with propagation around the azimuth (φ direction). The WS modes with m = 0 posses spatial structures and propagation patterns that are quite different to those of the WG modes. These modes propagate as circular wave-fronts that travel along the θ direction, converging at the extremes of the z-axis. These modes have been dubbed Whispering Longitudinal (WL) modes. WG modes are strongly confined to the equator whereas WL modes sample the entire surface of the sphere. They are not localized to any particular region, but do have energy density maxima at the poles. Modes with values of m between 1 and n-1 are essentially linear combinations of different order WG and WL modes.

IV. Lifting The Legendre Degeneracy

In a spherical system the θ and φ directions are essentially interchangeable due to the spherical symmetry, and hence there is no frequency difference between the degenerate modes. The key to breaking the degeneracy is the fact that the frequency of each mode within a mode family depends on different regions of the system to different extents. WG modes are essentially confined to the equatorial region, and are virtually insensitive to conditions at the poles. WL modes, on the other hand, converge at the extremes of the z-axis and are thus strongly influenced by polar conditions. Other members of the mode family will be influenced somewhere in between. The more WL-like the mode is (ie, the closer its value of m is to 0) the greater the effect of the polar conditions. This suggests that introducing some kind of perturbation that is cylindrically symmetric with respect to the polar axis will give rise to an orderly frequency separation of the degenerate mode families. Of course, the form of the separation will depend on specific characteristics of the perturbation.

Analytical solutions for all but a few perturbed systems do not exist, making it necessary to introduce numerical modeling as a means of discovering the mode structure. In the following dielectric loaded resonators the mode solution sets were found using finite element analysis in a cylindrical coordinate system [4].

V. Silica Loaded Cavity

One of the first physical systems in which WS modes have been investigated is a 50 mm inner diameter copper cavity loaded with a 38 mm diameter isotropic fused-silica sphere [6]. The dielectric is held in place by two 11 mm diameter Teflon supports located at the two poles. These supports constitute the perturbation required to lift the degeneracy of the mode families.

If the effect of the Teflon is ignored then the system can be solved analytically in the manner described above. The supported system was solved numerically. Figure 2 shows selected field density plots of TM_{61m} mode family through the r-z plane (cylindrical coordinates) for both the unsupported and supported resonator.

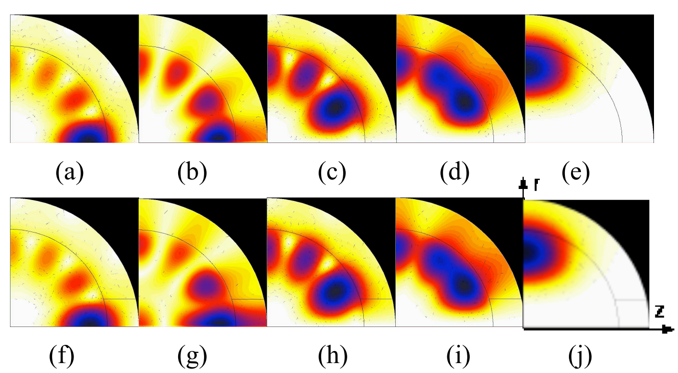

Figure 2: Modeled finite element electric field density plots of (a)-(e) the TM_{610}(WL), TM_{611}, TM_{612}, TM_{613} and TM_{616}(WG) modes in the (unphysical) unsupported system and the same modes (f)-(j) in the supported system. Each plot is one quarter of the r-z (meridian) plane, with the bottom axis corresponding to the axis of cylindrical symmetry. The support can be seen at the bottom right of figures (f)-(j). "Leakage" of field into the supports is the source of the frequency discrepancy between the two systems.

It can be seen from these plots that the "pulling" effect on the field of the two supports significantly alters the mode patterns of the WL like modes, while the WG mode pattern is virtually unchanged. Figures 3 (a)-(d) show predicted frequencies for both the unsupported and supported systems, as well as measurements of the fused-silica loaded cavity, for the first four TM_{n1m} mode families. Higher order mode families exhibit the same frequency splitting but the mode density is such that above n=5 successive mode families begin to overlap, making identification difficult.

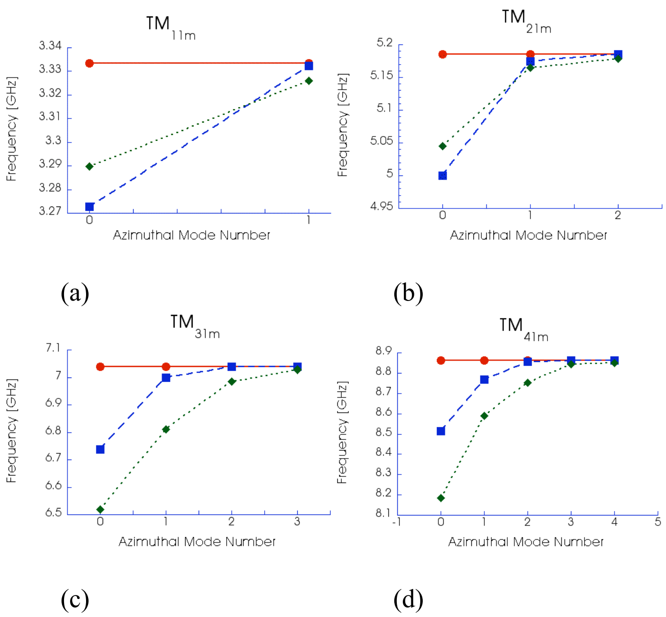

Figure 3: Predicted frequencies for the unsupported (solid line), supported (dashed line) and measured frequencies in the supported system (dotted line) for the (a) n=1, (b) n=2, (c) n=3 and (d) n=4 TM WS mode families.

The physical system required slightly different supports than those assumed in the finite element analysis, which was performed before construction of the system. Consequently the experimental frequency shift is different to that predicted by the numerical analysis. However this difference is really only one of scale. The form of the frequency shift is all that needs to be considered and that, for all practical purposes, appears to be the same. In all cases the measured shift of the WG mode is small, ranging from 6.27 to 12.2 MHz in the mode families studied. The WL modes experience a much larger frequency shift ranging from 130 to 680 MHz. The rest of the modes in each family experience a shift between the two extremes; each exhibiting successively smaller frequency shifts with increasing m.

VI. Sapphire Loaded Cavity

The same 50 mm diameter copper cavity was used to construct a second dielectrically-loaded resonator. A 30 mm sphere on a 10 mm diameter cylindrical support post, cut from a single piece of HEMEX sapphire, was used as the dielectric. Sapphire has an inherent uniaxial anisotropic permittivity. The relative permittivity is approximately 12 parallel to the anisotropy axis and 9 tangential to it. In this case the anisotropy axis was designed to coincide with that of the support post to ensure the resonator was cylindrically symmetric.

Frequency shifts of the degenerate mode families occur via two mechanisms. Firstly the inherent anisotropy of the sapphire implies that the effective path length parallel to the anisotropy axis is longer than that in the tangential direction, ie the sapphire sphere is effectively spheroidal. The second mechanism is similar to that in the fused-silica case. Polar field leakage into the support post tends to shift the frequencies of modes with low m.

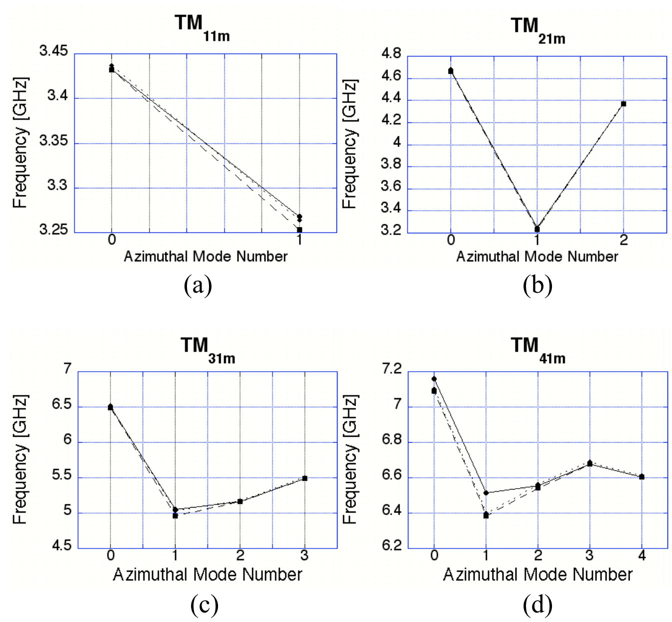

Figure 5: Calculated frequencies for the unsupported (solid line), supported (dashed line) and measured frequencies in the supported sapphire resonator (dotted line) for the (a) n=1, (b) n=2, (c) n=3 and (d) n=4 TM WS mode families.

The inherent anisotropy of the sapphire makes it difficult to solve even the unsupported resonator analytically. Both the supported and unsupported resonators had to be solved numerically. Figure 5 (a)-(d) show the predicted frequencies for the first 4 TM_{n1m} mode families for both the unsupported and supported system as well as the measured values. It can be seen from these plots that the frequency discrepancy between the two systems for each Legendre mode is small compared to the total frequency shift. This implies that the inherent anisotropy of the sapphire sphere is the dominant mechanism of frequency shift. It is also interesting to note that the form of the frequency deviations is very different to that of the fused-silica resonator.

In each case the predicted and measured frequencies differed on the order of tens of MHz, which is the order of the experimental error set by the dimensions of the resonator and the temperature offset between that assumed during modeling and ambient temperature at the time of measurement.

VII. Conclusions

Spherical systems operating on WS modes offer great potential as high Q microwave resonators, but in an isotropic system mode family frequency degeneracy makes it difficult to identify and utilize these modes. Introducing a perturbation into the spherical system is an easy and convenient way of frequency shifting each member of the otherwise frequency degenerate mode families. The perturbed system has been shown to be reliably modeled using finite element techniques in two loaded copper cavity resonators. Deviations from predicted frequency are at the same level as the experimental error set by the temperature offset and the dimensions of the resonators.

VII. References

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